B.Sc. (Part—III) Semester—VI Examination

MATHEMATICS (Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt this question once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative (1 mark each) :—

(i) Let A and B be non-empty subsets of a vector space V. Suppose that $A \subseteq B$, then :

(a) If B is LI, then so is A
(b) If B is LD, then so is A
(c) If A is LI, then so is B
(d) If B is generating set, then so is A

(ii) The vectors (a, b) and (c, d) are LD iff :

(a) ad–bc = 0
(b) ad–bc ≠ 0
(c) ad+bc = 0
(d) ad+bc ≠ 0

(iii) The kernel of a linear transformation $T:U \to V$ is a subspace of :

(a) V
(b) U
(c) U and V
(d) None of these

(iv) A square matrix is non-singular iff its row vectors are :

(a) LD
(b) LI
(c) LI and LD
(d) None of these

(v) Annihilator of W, A(W) is a subspace of :

(a) $\hat{V}$
(b) W
(c) V
(d) None of these

(vi) If T be a linear map on R^2 which is represented in the standard basis by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Then the eigen values of T are :

(a) ±1
(b) ± i
(c) ±2
(d) None of these

(vii) If u and v be vectors in an inner product space such that $\|u+v\| = 8$, $\|u-v\| = 6$ and $\|u\|=7$ then the value of $\|v\|$ is :

(a) $\sqrt{51}$
(b) $\sqrt{2}$
(c) 1
(d) 2

(viii) In an inner product space V over F the inequality $|(u, v)| \leq \|u\| \, \|v\| \; \forall \; u, v \in V$ is known as :

(a) Triangular inequality (b) Bessel's inequality

(c) Cauchy-Schwarz inequality (d) None of these

(ix) If A is any submodule of an R-module M, then the zero element of the quotient group M/A is :

(a) A (b) {o}

(c) M (d) None of these

(x) R-module homomorphism is a linear transformation of vector spaces if :

(a) R with unit element (b) R is commutative

(c) R is field (d) None of these

UNIT—I

2. (a) Let R^+ be the set of all positive real numbers. Define the operation of addition $\oplus$ and scalar multiplication $\otimes$ as follows :

$u \oplus v = uv, \quad \forall \; u, v \in R^+$

and $\alpha \otimes u = u^{\alpha}, \quad \forall \; u \in R^+, \; \alpha \in R$

Prove that R^+ is a real vector space. 4

(b) Prove that an arbitrary intersection of Subspaces of a vector space is again a subspace. 3

(c) If x, y, z are LI vectors of a vector space V, then prove that x+y, y+z, z+x are LI. 3

3. (p) Show that the ordered set S = {(1, 1, 2), (1, –1, 1), (1, 3, 3), (–1, 3, 0)} is LD and locate one of the vectors that belongs to the span of previous one. Find also the largest LI subset whose span is equal [S]. 5

(q) Define subspace of a vector space and let U, W be subspaces of a vector space V(F). Prove that $U \cup W$ is a subspace of V iff $U \subseteq W$ or $W \subseteq U$. 1+4

UNIT—II

4. (a) Let U, V be vector spaces over a field F and $T : U \to V$ be a linear map, then prove that :

(i) $T(\bar{o}) = \bar{o}$,

(ii) $T(-u) = -T(u), \quad \forall \; u \in U$

(iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots\ldots + \alpha_n T(u_n)$, $\forall \; u_i \in U, \alpha_i \in F, 1 \leq i \leq n$ and $n \in N$.

1+1+3

(b) Find the matrix of the linear map $T : V_2 \to V_3$ defined by $T(x, y) = (-x + 2y, y, -3x + 3y)$ related to the bases $B_1 = \{(1, 2), (-2, 1)\}$ and $B_2 \{(-1, 0, 2), (1, 2, 3), (1, -1, 1)\}$. 5

5. (p) A linear transformation T is completely determined by its values on the elements of a basis. Precisely, if B = $\{u_1, u_2, \ldots\ldots\ u_n\}$ is a basis for U and v_1, v_2, ...v_n be n vectors (not neceessarity distinct) in V. Then prove that there exists a unique linear transformation $T : U \rightarrow V$ such that $T(u_i) = v_i$, for i =1, 2, ..., n. 5

(q) Find the range, kernel, rank and nullity of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and verify Rank -nullity theorem. 5

UNIT—III

6. (a) Let V be the finite dimensional vector space over F. Then prove that $V \approx \hat{\hat{V}}$. 5

(b) The element $\lambda \in F$ is a CR of $T \in L(V)$ iff for some $V(\neq 0) \in V$, $T_V = \lambda_V$. Prove this. 3

(c) Define eigen values and eigen vectors of a matrix. 2

7. (p) If W be a subspace of a finite dimensional vector space V, then prove that A(A(w)) =W. 5

(q) If K_λ is an eigen space, then prove that K_λ is subspace of vector space V. 3

(r) Let $\lambda \neq 0$ be an eigen value of an invertible LT, $T \in L(V)$. Then show that λ^{-1} is an eigen value of T^{-1}. 2

UNIT—IV

8. (a) In c^2 define, for $u = (\alpha_1, \alpha_2)$ and $v = (\beta_1, \beta_2)$, $(u, v) = 2\alpha_1\overline{\beta}_1 + \alpha_1\overline{\beta}_2 + \alpha_2\overline{\beta}_1 + \alpha_2\overline{\beta}_2$.

Show that this defines an inner product on c^2. 5

(b) Find the orthonormal basis of $P_2[-1, 1]$ starting from the basis $\{1, x, x^2\}$ using the inner product defined by : $(f, g) = \int_{-1}^{1} f(x).g(x)\, dx$. 5

9. (p) Define othogonal set and let $\{x_1, x_2, \ldots\ldots, x_n\}$ be an orthogonal set. Then prove that

$\|x_1 + x_2 + \ldots\ldots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \ldots\ldots + \|x_n\|^2$. 1+4

(p) If $\{w_1, \ldots\ldots w_m\}$ is an orthonormal set in V, then prove that :

$\sum_{i=1}^{m} |(w_i, v)|^2 \leq \|v\|^2$, for any $v \in V$. 5

UNIT—V

10. (a) Let M be an R-module. Then prove that :

(i) $r0 = 0, \ \forall \ r \in R$

(ii) $-(ra) = r(-a) = (-r)a, \ \forall \ r \in R$ and $a \in M$. 4

(b) If λ is a left ideal of R and if M is an R-module, then show that for $m \in M$, $\lambda_m = \{xm \mid x \in \lambda\}$ is a submodule of M. 3

(c) If $T : M \rightarrow H$ be an R-module homomorphism, then prove that Ker T is a submodule of M. 3

11. (p) Define unital R-module and let A be a submodule of unital R-module M, then prove that M|A is also a unital R-module. 1+4

(q) Let M be an R-module. If H and K are submodules of M with $K \subset H$, then prove that $\frac{M}{H} \approx \frac{M|K}{H|K}$. 5